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General integral properties of mixing layers

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ABSTRACT

If the usual integrals defining displacement and momentum thicknesses are used for mixing layers with a change in the range of integration, they diverge in general. Consequently, the familiar von Karman equation is not generally applicable to a mixing layer. Previous works used the device of dividing the layer into two sub-layers and using one integral for each separately with a continuity condition on shear stress. Suitable definitions of displacement and other thicknesses are given here, and a momentum integral relation is obtained. Its form is different from von Karman's equation. The formulation given here is applicable to incompressible and compressible fluids, laminar and turbulent flows and to two-dimensional and axisymmetric flows.

1. INTRODUCTION

Mixing between two largely inviscid streams is of interest in problems of base flows, mixing of chemicals, meteorology and studies on the structure of turbulence.

Classical definitions of displacement and other thicknesses are designed for boundary-layer type of flows which are confined on one side by a solid wall. They cannot be applied with a simple change of range of integration to a layer between two streams, as the integrals used in the definitions diverge in general. Consequently, the familiar von Karman momentum integral equation is not applicable to such a layer as a whole. However, it is possible to divide the layer into a top part and a bottom part and to apply the usual relation to the parts separately and to use continuity of shear stress. Such a device was used for example, by Lock (1951) who studied the mixing between two incompressible fluids. When the fluids have different properties, there is a well defined interface which can be used as a natural boundary. Kubota and Dewey (1964) used this device when the properties varied continuously. Quite apart from the arbitrariness of the boundary between the two parts in the absence of a discontinuity in material properties, the number of unknowns (thickness parameters on either side) and the number of conditions (one equation for each sub-layer and continuity conditions) increase thus adding complexity to analysis.

This brief paper extends the definitions of the thicknesses for mixing layers and obtains corresponding integral relations.

The formulation given here is valid for incompressible and compressible fluids, steady laminar and stationary turbulent flows, and for two-dimensional and axisymmetric flows. Pressure gradient may be non-zero. In axisymmetric flows, it is assumed that the thickness of the layer is small in comparison with the distance of the layer from the axis of symmetry.

It is of course assumed that boundary-layer approximations are valid.

The motion of the layer is governed by the familiar continuity and momentum equations, namely,

$$(\rho r^k u)_x + (\rho r^k v)_y = 0, \tag{1}$$

$$\rho(u u_x + v u_y) = -p_x + \tau_y, \tag{2}$$

and the boundary conditions are

$$y \to \infty: \quad u \to U_1(x), \quad \rho \to \rho_1(x), \quad \tau \to 0,$$

$$y \to -\infty: \quad u \to U_2(x), \quad \rho \to \rho_2(x), \quad \tau \to 0. \tag{3}$$

Constancy of pressure across the layer requires that $\rho_1 d\,(U_1{}^2)/dx - \rho_2 d\,(U_2{}^2)\,/\,dx$ is zero.

A third boundary condition on normal velocities at large $|y|$ or displacements of streamlines is required to make the solution unique. This condition can in principle be obtained by higher-order analysis of the inviscid streams. Usual treatment is to rely on the Prandtl's transposition theorem (Rosenhead, 1963) and to obtain a standard solution by requiring that x-axis is a stream-line. Other solutions can be generated from the standard solution by using the theorem. (In symmetrical flows, an additional condition can be readily obtained by using symmetry properties). In addition, suitable initial conditions are needed for the solution, but they do not concern us here.

u, v are the velocity components in the usual boundary-layer type of x, y coordinates, x-axis lying within the layer; ρ, $p(x)$ and τ are density, pressure and shear stress. $r(x)$ is the radial distance from the axis of symmetry in the axi-symmetric case, and k is zero or one depending on whether the flow is two-dimensional or axisymmetric. Suffixes x and y denote partial differentiation, and suffixes 1 and 2 the values at the two edges.

When the fluid is incompressible and the flow is turbulent, the quantities refer to mean values and τ is the total shear stress. $p(x)$ refers to mean pressure at the edges and the normal stress terms in the momentum equation, which are sometimes considered by some authors, are absent in the approximation considered here (Rotta, 1962, p.13).

In the case of turbulent mixing layer of a compressible fluid, v is the sum of the mean normal velocity and a term involving fluctuations of density and normal velocity (*i.e.* $\overline{\rho' v'}/\rho$), so that equation (1) continues to remain valid. (Schubauer and Tchen, 1961, p. 17)

For the present purpose, it is not necessary to consider the equation of state, internal energy or any model of turbulence.

2. Definition

The displacement thickness δ^* and thickness δ_q of any extensive quantity q are defined by

$$\int_{y_2}^{y_1} \rho u dy - \rho_1 U_1 (y_1 - \delta^*) + \rho_2 U_2 (y_2 - \delta^*) \to 0, \tag{4}$$

$$\int_{y_2}^{y_1} \rho u q dy - \rho_1 U_1 q_1 (y_1 - \delta^* - \delta_q) + \rho_2 U_2 q_2 (y_2 - \delta^* - \delta_q) \to 0, \tag{5}$$

as $y_1 \to \infty$, $y_2 \to -\infty$. It is assumed that $q \to q_1(x)$ as $y \to \infty$ and $q \to q_2(x)$ as $y \to -\infty$.

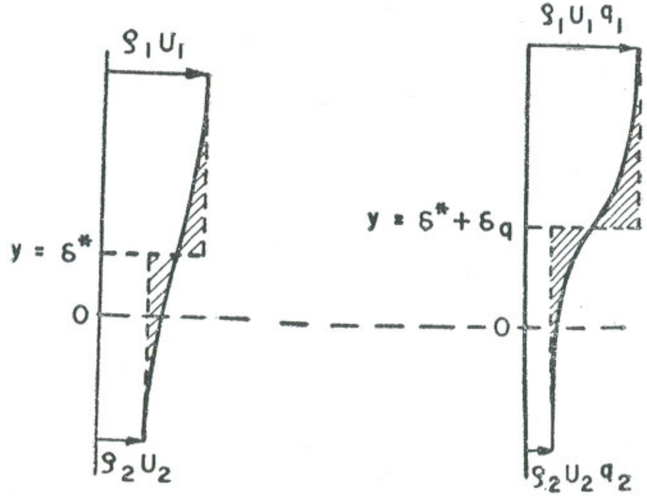

Fig. 1. Interpretation of δ^* and δ_p

As shown in Fig. 1, δ^* can be interpreted as the location of an infinitesimal layer which has the same mass flux as the given layer and, $\delta^* + \delta_q$ as the location of another infinitesimal layer which has the same flux of of q as the given layer. As δ^* and δ_q describe locations of infinitesimal layers, they can be negative despite the contrary suggestion in the terminology of the thickness. Also, the value of δ^* depends on the location of the x-axis, but δ_q is invarient under translation of the x-axis.

It is customary to relate δ^* to the displacement of a distant streamline. If x-axis is along a streamline and if the initial profiles on both sides of the streamline are uniform, the displacements $\delta_1{}^*$ and $\delta_2{}^*$ of the two streams away from the x-axis would be

$$\rho_1 U_1 \delta_1{}^* = \int_0^{y_1} (\rho_1 U_1 - \rho u) dy, \qquad \rho_2 U_2 \delta_2{}^* = \int_{y_2}^{0} (\rho_2 U_2 - \rho u) dy, \tag{4a}$$

so that

$$(\rho_1 U_1 - \rho_2 U_2)\delta^* = \rho_1 U_1 \delta_1{}^* + \rho_2 U_2 \delta_2{}^*. \tag{4b}$$

So δ^* is a weighted average of the displacements of the distant streamlines on the two sides. In the special case when U_2 is zero, δ^* differs from the displacement of a distant streamline on the top side. (Note that in this case the distant streamlines on the lower sides come from large distances making $\delta_2{}^*$ infinite).

Clearly, relation (5) can be used to define momentum thickness $\theta(q=u)$, energy thickness, enthalpy thickness, concentration thickness and Reynolds stress thickness etc.

If U_2 is zero, and $u(y)$ is zero for $y \leqslant 0$, the above definitions reduce to the usual definitions for a boundary-layer.

The following expressions for δ^* and δ_q can now be readily obtained from (4) and (5).

$$\delta^* = \int_0^{\infty} \frac{\rho_1 U_1 - \rho u}{\rho_1 U_1 - \rho_2 U_2} dy + \int_{-\infty}^{0} \frac{\rho_2 U_2 - \rho u}{\rho_1 U_1 - \rho_2 U_2} dy, \tag{6}$$

$$= \lim_{h \to \infty} \int_{-h}^{h} \frac{\rho_1 U_1 + \rho_2 U_2 - 2\rho u}{2(\rho_1 U_1 - \rho_2 U_2)} dy, \tag{7}$$

$$\delta_q = \int_{-\infty}^{\infty} \frac{\rho_1 U_1 q_1 - \rho u q}{\rho_1 U_1 q_1 - \rho_2 U_2 q_2} - \frac{\rho_1 U_1 - \rho u}{\rho_1 U_1 - \rho_2 U_2}\, dy, \tag{8}$$

$$= \int_{-\infty}^{\infty} \frac{\rho_2 U_2 q_2 - \rho u q}{\rho_1 U_1 q_1 - \rho_2 U_2 q_2} - \frac{\rho_2 U_2 - \rho u}{\rho_1 U_1 - \rho_2 U_2}\, dy, \tag{9}$$

$$= \int_{-\infty}^{\infty} \frac{\rho_1 U_1 q_1 + \rho_2 U_2 q_2 - 2\rho u q}{2(\rho_1 U_1 q_1 - \rho_2 U_2 q_2)} - \frac{\rho_1 U_1 + \rho_2 U_2 - 2\rho u}{2(\rho_1 U_1 - \rho_2 U_2)}\, dy. \tag{10}$$

The above expressions may become singular as $\rho_1 U_1 \rightarrow \rho_2 U_2$ and $\rho_1 U_1 q_1 \rightarrow \rho_2 U_2 q_2$, if the numerators do not approach zero sufficiently rapidly.

Alternate definitions like

$$\delta^{*\prime} = \int_0^{\infty} (1 - \frac{\rho u}{\rho_1 U_1})\, dy + \int_{-\infty}^{0} (1 - \frac{\rho u}{\rho_2 U_2})\, dy \tag{11}$$

have the disadvantage of becoming singular when the fluid is at rest on one side of the layer, i. e U_2 is zero. Since this case is of considerable interest and has been extensively investigated, it is desirable that the integrals used in the definition do not diverge for the case $U_2 = 0$, $U_1 \neq 0$.

Clearly, the preceeding definitions are not designed for wakes $(U_1 = U_2 \neq 0$, etc.) or jets in stationary surroundings $(U_1 = U_2 = 0$, etc.) (Appendix shows how greater generality can be attained).

3. Integral Relations

Attention is focussed on obtaining mass and momentum integrals. Other integrals for kinetic energy, enthalpy, concentration, Reynolds stress etc. can be obtained by a similar reasoning.

The outer inviscid flow $(U, V, \bar{P})$ is given by

$$U \sim U_i(x), \quad V \sim V_{iy}\, y, \quad \rho \sim \rho_i.$$

i is one for upper edge $(y>0)$ and two for the lower edge $(y<0)$. $V_{iy}(x)$ is the value of the normal derivative at the axis y = 0, where it is assumed that $V_i = 0$. Mass and momentum equations for inviscid flow are

$$(\rho_i r^k U_i)_x - \rho_i r^k V_{iy} = 0, \tag{13}$$

$$\rho_i U_i U_{ix} = -P_x \tag{14}$$

Here repeated indices do not imply summation.

On the basis of known properties of boundary-layer equations, we assume that

$$u = U_i(x) + 0(y^n), \quad v = V_{iy} y + v_i + o(1),$$

$$= \rho_i(x) + \tilde{\rho}_i / y + o(y^{-1}), \text{ for any } n, \text{ as } |y| \rightarrow \infty, \tag{15}$$

so that

$$v = \rho_i V_{iy} y + \rho_i v_i + \tilde{\rho}_i V_{iy} + o(1). \tag{16}$$

Although there is no clear general indication about the behaviour of ρ from boundary-layer flows, we will assume for simplicity $\tilde{\rho}_i = 0$ (If it is not zero, the term $\tilde{\rho}_i V_{iy}$ can be added to the equations obtained below.).

To obtain the mass integral, we first get from (1) and (13)

$$[r^k(\rho_i U_i - \rho u)]_x + [r^k(\rho_i V_{iy} y - \rho v)]_y = 0. \tag{17}$$

The above equation is integrated across the layer with $i=1$ for $y \geqslant 0$ and $i=2$ for $y<0$. Use of (6) and (16) then yields the mass integral relation in the form

$$\frac{1}{r^k}\frac{d}{dx}\left[r^k(\rho_1 U_1-\rho_2 U_2)\delta^*\right]-(\rho_1 v_1-\rho_2 v_2)=0. \tag{18}$$

While the rate of change of displacement thickness can be simply related to the slope of a streamline at the outer edge in a boundary-layer, the relationship is more complex in a mixing layer. This is to be expected for if the fluid is at rest on one side of the layer, the entrained fluid comes into the layer in a normal direction and the slope of the streamline there is infinite.

To obtain the momentum integral, we first write the expression for momentum thickness θ by taking $q=u$ and using (5) as

$$(\rho_1 U_1^2-\rho_2 U_2^2)(\delta^*+\theta)=\int_0^\infty(\rho_1 U_1^2-\rho u^2)\,dy+\int_{-\infty}^\infty \rho_2 U_2^2-\rho u^2)\,dy \tag{19}$$

We then obtain, by multiplying (1), (2), (13) and (14) by $-u$, $-r^k, v_i$ and r^k, respectively and adding them,

$$[r^k(\rho_i U_i^2-\rho u^2)]_x+[r^k\{(\rho_i U_i V_{Iy} y-\rho uv)+\tau\}\tau]_y=0. \tag{20}$$

Integration across the layer with $i=1$ for $y \geqslant 0$ and $i=2$ for $y<0$ gives

$$\frac{1}{r^k}\frac{d}{dx}\left[r^k(\rho_1 U_1^2-\rho_2 U_2^2)(\theta+\delta^*)\right]-(\rho_1 U_1 v_1-\rho_2 U_2 v_2)=0. \tag{21}$$

Here (3), (15), (16), and (19) have been used.

It should be emphasised that integral relations (18) and (21) are sufficiently general to permit the presence of pressure gradient, variation of properties etc. Also, v_i represents the excess normal velocity induced by the layer at the edges and is not in general equal to the normal velocity of a fluid particle at the edge of the layer.

Elimination of $v_i (i=1$ or $2)$ from (18) and (21) yields

$$\left\{\rho_1 U_1+\rho_2 U_2+\frac{\rho_1-\rho_2}{U_1-U_2}U_1 U_2\right\}\frac{1}{r^k}\frac{d(r^k\theta)}{dx}+\frac{\theta}{(U_1-U_2}\left\{\frac{U_1^2 d\rho_1}{dx}-\frac{U_2^2 d\rho_2}{dx}\right\}$$

$$+\frac{\rho_i U_i}{r^k}\frac{d}{dx}(r^k\delta^*)+\delta^*\frac{d(\rho_i U_i)}{dx}-\rho_i v_i=0 \tag{22}$$

The above equation may be compared with the classical von Karman equation (say. Rosenhead, 1963, chapter 5, equation (42). The last term arising from entrainment is similar to the suction term in the classical equation and it has to be obtained from the third boundary condition discussed earlier.

Simpler froms of (22) are given below for particular cases.

(a) *Incompressible fluids:* Conside: two streams of homogeneous incompressible fluids of equal density (*i. e* $\rho=\rho_1=\rho_2$). Then (22) can be written as

$$(U_1+U_2)\frac{1}{r^k}\frac{d}{dx}(r^k\theta)+\frac{U_i}{r^k}\frac{d(r^k\delta^*)}{dx}+\delta^*\frac{dU_i}{dx}-v_i=0 \tag{22 a}$$

(b) *Constant outer stream conditions*: This case of zero pressure gradient is also of special interest, (ρ_i and U_i do not change with x). Relation (22) then becomes

$$\{\rho_1 U_1+\rho_2 U_2+\frac{\rho_1-\rho_2}{U_1-U_2}U_1 U_2\}\frac{1}{r^k}\frac{(dr^k\theta)}{dx}+\frac{\rho_i U_i}{r^k}\frac{d(r^k\delta^*)}{dx}-\rho_i v_i=0 \tag{22b}$$

(c) *When fluid is at rest on one side:* This case is of considerable interest. Say, U_2 is zero. Constancy of pressure requires that U_1 does not change with x. Equation (22) then simplifies to

$$\frac{1}{r^k}\frac{d(r^k\theta)}{dx}+\frac{\theta}{\rho_1}\frac{d\rho_1}{dx}+\frac{\rho_l U_l}{\rho_1 U_1}\frac{1}{r^k}\frac{d(r^k\delta^*)}{dx}\frac{\delta^* U_l}{\rho_1 U_1}\frac{d\rho_1}{dx}-\frac{\rho_l v_l}{\rho_1 U_1}=0. \qquad (22c)$$

(Other equations like equation of state may be used to show that $d\rho_1/dx$ is zero.)

(*d*) *When x-axis is a streamline* : This condition is sometimes used in mixing layer analysis. Integration of (17) with the condition that v is zero at $y=0$ leads to

$$\frac{1}{r^k}\frac{d}{dx}\left[\int_0^{y_l} r^k(\rho_l U_l-\rho u)dy\right]-\rho_l v_l=0. \qquad (18a)$$

Elimination of $\rho_l v_l$ from (18a) and (21) and use of (19) gives

$$r^k\left\{\int_0^{\infty}\rho u(U_1-u)dy+\int_{-\infty}^{0}\rho u(U_2-u)dy\right\}=\text{constant}.$$

The above relation may be compared with the special case given by Lock (1951, p.45).

Integral relations (18), (21) and (22) can also be obtained by corresponding control volume arguments.

The preceeding momentum integral relation can be used in several ways. One is to check the two-dimensionality or axisymmetric nature of flows observed in experiments. The other is to assess errors in a numerical calculation, (*e. g.* say one using implicit finite-difference marching scheme.) Still another is in the prediction of gross quantities by assuming a family of profiles and possibly some integral hypothesis of turbulence like Head's entrainment hypothesis of turbulence like Head's entrainment hypothesis. In this connection, it could provide a way of accounting for the effect of initial boundary-layer conditions and possibly non-similar profiles in supersonic base flows.

4. Other integral parameters

Two integral parameters are frequently used in base flow models following their introduction by Korst (1954 and 1956). Here we will give somewhat more general definitions of these parameters and relate them to δ^* and θ. (i.e. assumptions of constant total temperature, similarity of profile, and use of certain coordinates are not made.) We will restrict attention to the case (*c*). As shown in Figure 2, the mass balance of control volume *ABCD* gives

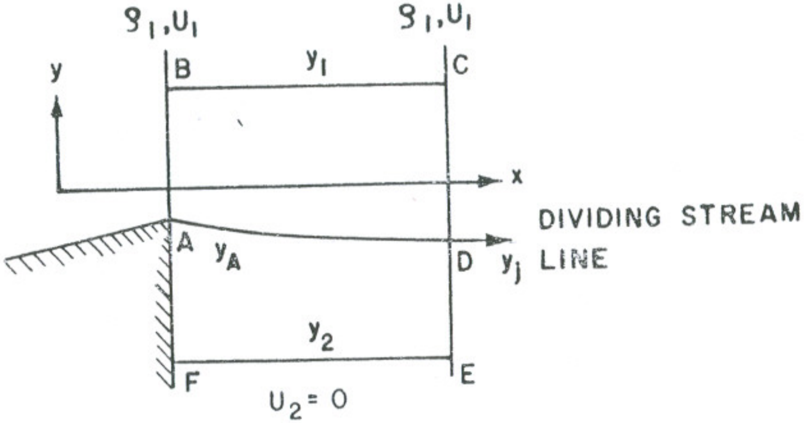

Fig. 2. Location of dividing streamline

$$(2\pi r)^k\int_{y_j}^{y_1}\rho u dy=(2\pi r_o)^k\int_{y_A}^{y_1}\rho u dy+\dot{m}_1 \qquad (23)$$

Suffix zero indicates initial condition and $\dot{m}_1$ is the mass flux through side *BC*. Here y_j is the coordinate of the streamline dividing the fluid of the stream from that which was originally at rest. Similarly the momentum balance of control volume FBCE, over which pressure is constant, gives

$$(2\pi r)^k\int_{y_2}^{y_1}\rho u^2 dy=U_1\dot{m}_1+(2\pi r_o)^k\left[U_1\int_{y_A}^{y_1}\rho u dy-\rho_1 U_1^2\theta_o\right]_0 \qquad (24)$$

Here suffix 0 indicates initial values. When y_2 is taken sufficiently away from the layer, the momentum flux of mass entering through FF is zero. Elimination of m_1 from (22) and (24) results into.

$$r^k \int_{y_2}^{y_1} \rho u^2 dy = r^k U_1 \int_{y_j}^{y_1} \rho u dy - r_0^k \rho_1 U_1^2 \theta_0. \tag{25}$$

The above relation can be used to evaluate y_j the location of the dividing streamline if $u(y)$, $\rho(y)$, r, r_0 and θ_0 are known. However as y_1 increases both the integrals diverge and evaluation of y_j becomes awkward. A more convenient form can be obtained by using (4) and (5).

$$r^k \rho_1 U_1^2 (y_1 - \delta^* - \theta) = r^k U_1 \left[\rho_1 U_1 (y_1 - \delta^*) - \int_{-\infty}^{y_j} \rho u dy \right] - r_0^k \rho_1 U_1^2 \theta_1,$$

or

$$\int_{-\infty}^{y_j} \frac{\rho u}{\rho_1 U_1} dy = \theta - \frac{(r_0)^k}{r} \theta_0. \tag{26}$$

The equations (25) and (26) may be compared with equation (31) of Charwat (1970).

Since the term of the left side is simply the mass flux of entrained fluid from the lower side, the above equation can also be obtained by equation (22c).

The second integral parameter introduced by Korst was the location of an "intrinsic" coordinate system. We define this location by y_m given by

$$(2\pi r)^k \int_{-\infty}^{y_j} \rho u^2 dy = (2\pi r_0)^k \int_{y_A}^{y_1} \rho u^2 dy + (2\pi r)^k \rho_1 U_1^2 y_m. \tag{27}$$

(The sign of y_m in the above is different from the sign used by Korst, as it refers here to the location of the "intrinsic" axis in the given coordinates, while it referred in Korst's work to the location of a given coordinate axis in the "intrinsic" system. The definitions (6) and (8) immediately give the following simple relation

$$r^k (y_m + \theta + \delta^*) = r_0^k (y_A + \theta_0 + \delta_0^*). \tag{28}$$

Thus changes of y_m indicate change of $\delta^* + \theta$ and not only momentum thickness.

Acknowledgements

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Appendix

It is possible to attain greater generality at a certain price. For instance, we can compare the mass flux and flux of q between y_1 and y_2 with corresponding fluxes of an inviscid flow with a discontinuity at a chosen location y_3 and obtain length scales and δ^* and δ_q in terms of chosen scales ρ_r, U_r and q_r of density, velocity and q. We accordingly define $\bar{\delta}^*$ and $\bar{\delta}_q$ by

$$\int_{y_2}^{y_1} \rho u dy - \rho_1 U_1 (y_1 - y_3) + \rho_2 U_2 (y_2 - y_3) + \rho_r U_r \bar{\delta}^* \to 0, \tag{4a}$$

$$\int_{y_2}^{y_1} \rho u q dy - \rho_1 U_1 q_1 (y_1 - y_3) + \rho_2 U_2 q_2 (y_2 - y_3) + \rho_r U_r q_r (\bar{\delta}^* + \bar{\delta}_q) \to 0, \tag{5a}$$

as $y_1 \to \infty$, and $y_2 \to -\infty$.

If the flow has an axis of symmetry, as in a symmetric jet or wake, the location y_3 can be taken to coincide with the axis. If the location y_3 is taken to be zero, the scales $\bar{\delta}^*$ and $\bar{\delta}_q$ generally depend on the coordinate axes. Since mixing layers usually do not have an axis of symmetry of mean velocity profile, we may opt for some preferred location y_3(*e.g.* maxima of vorticity, vanishing of normal velocity etc.), or we may choose to make $\overline{\delta_q}$ independent of y_3.

From (4a) and (5a), we get

$$\int_{y_2}^{y_1} \rho u(q-q_r)dy - \rho_1 U_1 y_1(q_1-q_r) + \rho_2 U_2 y_2(q_2-q_r)$$

$$+[y_3\{\rho_1 U_1(q_1-q_r)-\rho_2 U_2(q_2-q_r)\}]+\rho_r U_r q_r \bar{\delta}_q \to 0, \tag{29}$$

as $y_1 \to \infty$, and $y_2 \to -\infty$.

If the reference quantities ρ_r, U_r, and q_r are independent of y_3, the last bracket contains all the terms which may depend on y_3. If $\overline{\delta_q}$ is to be independent of y_3,

$$q_r(\rho_1 U_1 - \rho_2 U_2) - (\rho_1 U_1 q_1 - \rho_2 U_2 q_2) = 0 \tag{30}$$

This condition is trivially satisfied in flows in which $\rho_1 = \rho_2$, $U_1 = U_2$, $q_1 = q_2$, as in symmetrical jets and wakes.

Definitions (4) and (5) are special case of (4*a*) and (5*a*), with y_3 taken as zero, and

$$\rho_r U_r = (\rho_1 U_1 - \rho_2 U_2), \qquad \rho_r U_r q_r = (\rho_1 U_1 q_1 - \rho_2 U_2 q_2), \tag{31}$$

and hence the condition (30) is satisfied.

Apart from possible dependence on y_3, the definitions (4*a*) and (5*a*) do not lead to a simple interpretation of the type shown in Fig. 1.

Clearly, the definitions (4*a*) and (5*a*) lead only to minor modifications of (18) and (21) where $(\rho_1 U_1 - \rho_2 U_2)\delta^*$ and $(\rho_1 U_1^2 - \rho_2 U_2^2)(\delta^* + \theta)$ are replaced by $\rho_r U_r \delta^*$ and $\rho_r U_r^2(\bar{\delta}^* + \bar{\theta})$.

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